5. KARAKTERIZACIJA SISTEMA U STACIONARNOM STANJU

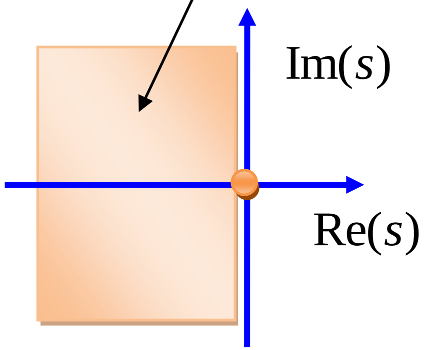
Prelazni režim

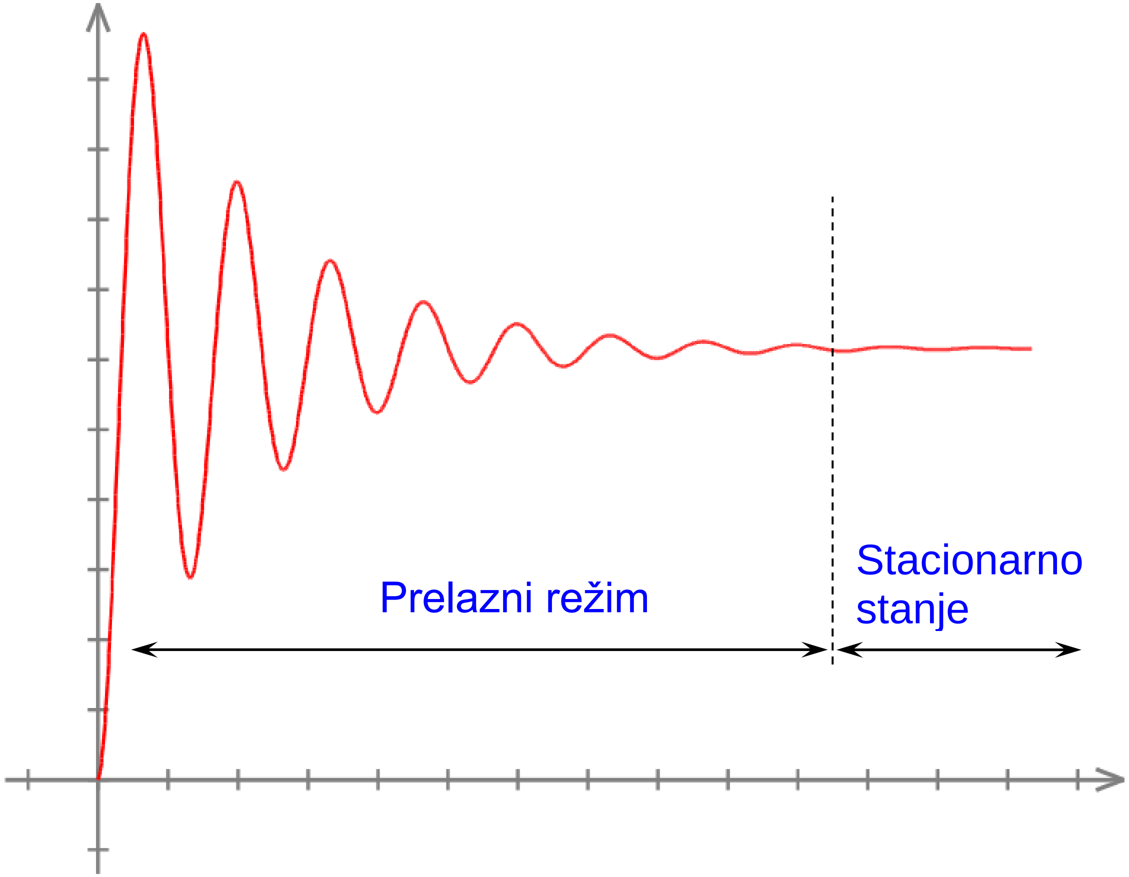
Stacionarno stanje

$$x(\infty) = \lim_{t \to \infty} x(t)$$

Druga granična teorema Laplace-ove transformacije:

$$x(\infty) = \lim_{s \to 0} s \boxed{X(s)}$$

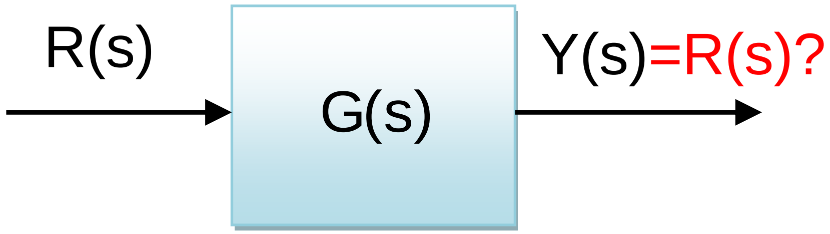




5.1. GREŠKA RADA SISTEMA BEZ POVRATNE SPREGE U STACIONARNOM STANJU

Greška rada sistema iznosi:

$$E(s) \triangleq R(s) - Y(s) = R(s) - G(s)R(s)$$
$$= \left[1 - G(s)\right] R(s)$$



Greška zavisi od:

1. funkcije prenosa $G(s)$ (tj. strukture i parametara sistema)
2. ulaznog signala

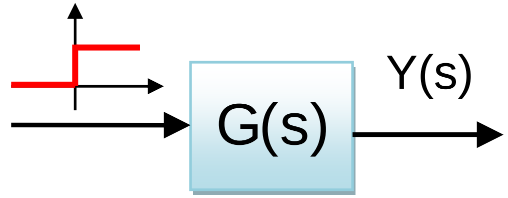
$E(s) = 0 \quad \Rightarrow \quad e(t) = 0$ je nemoguće postići u svakom trenutku

Cilj: greška u stacionarnom stanju (statička greška) je nula

$$e(\infty) = 0$$

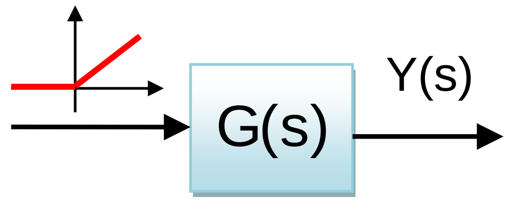
Tipovi statičkih grešaka e_S u odnosu na vrstu pobude

Poziciona statička greška: pobuda h(t)



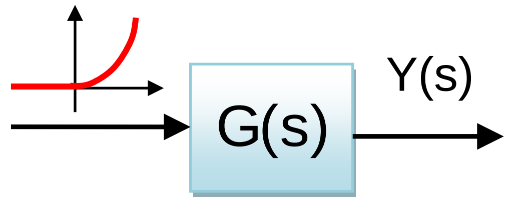
$$e_{SP} = \lim_{t \to \infty} \big(h(t) - y(t) \big)$$

Brzinska statička greška: pobuda r(t) = t h(t)



$$e_{SV} = \lim_{t \to \infty} \big(th(t) - y(t) \big)$$

Statička greška ubrzanja: pobuda t^2 h(t)/2



$$e_{SA} = \lim_{t \to \infty} \left(\frac{1}{2} t^2 h(t) - y(t) \right)$$

Određivanje pozicione statičke greške

Uslov: $G(s)$ ima polove striktno u levoj poluravni (važi II gr. teorema Laplasa).

Greška:

$$e_{SP} = \lim_{t\to\infty} e_P(t) = \lim_{s\to 0} sE_P(s)$$

$$= \lim_{s\to 0} s\left(\frac{1}{s} - G(s)\frac{1}{s}\right)$$

$$= 1 - \lim_{s\to 0} G(s)$$

$$= 1 - G(0)$$

$$= 1 - K_P$$

$$e_{SP} = \begin{cases} 0, & K_P = 1 \\ -\infty, & K_P \neq 1 \end{cases}$$

$$G(0) = 1 \quad \Rightarrow \quad e_{SP} = 0$$

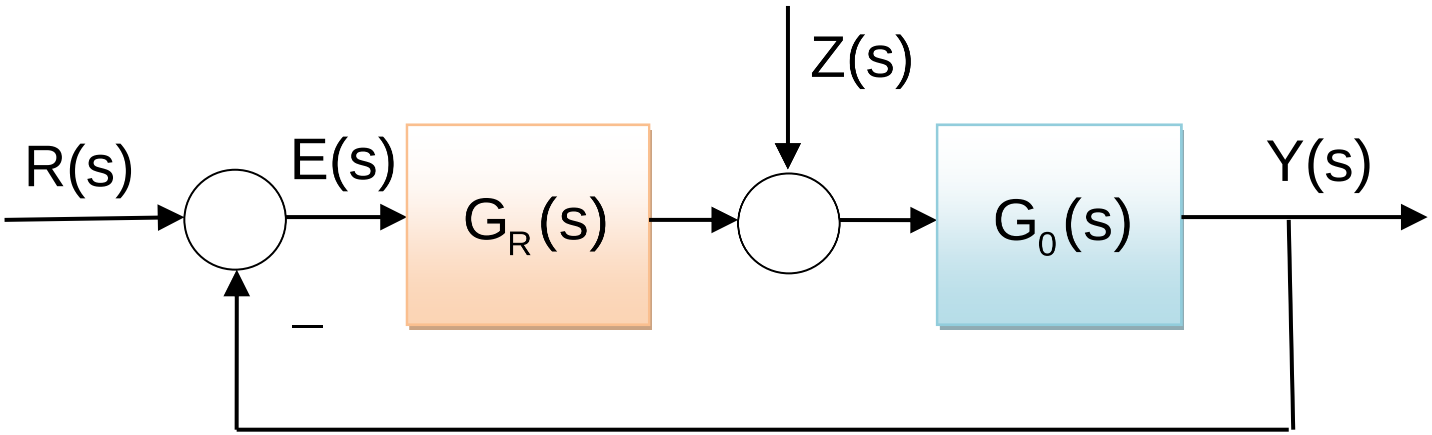
$K_P = G(0)$ - poziciono pojačanje sistema

$G(0) = 1$ može se postići određenom kalibracijom sistema.

$G(0) = 1$ je veoma teško održati u slučaju kada:

- na sistem deliju različiti poremećaji ili u
- menjaju se parametri i/ili struktura sistema

5.2. KARAKTERIZACIJA SISTEMA U ZATVORENOJ POVRATNOJ SPREZI U STACIONARNOM STANJU



$$E(s) = R(s) - Y(s) = R(s) - G_0(s)\left[Z(s) + G_R(s)E(s)\right]$$

$$E(s)\left[1 + G_0(s)G_R(s)\right] = R(s) - G_0(s)Z(s)$$

$$E(s) = \frac{1}{1+G(s)}R(s) + \frac{-G_0(s)}{1+G(s)}Z(s)$$

$G(s) = G_0(s)G_R(s)$ (funkcija povratnog prenosa)

Faktor pojačanja i red astatizma objekta, regulatora i funkcije povratnog prenosa

$$G_0(s) = K_O \frac{P_0(s)}{s^{r_0} Q_0(s)}, \qquad P_0(0) = Q_0(0) = 1,$$

K_0 - faktor pojačanja objekta G_0

r_0 - redi astatizma objekta G_0

$$G_R(s) = K_R \frac{P_R(s)}{s^{r_R} Q_R(s)}, \qquad P_R(0) = Q_R(0) = 1,$$

K_R - faktor pojačanja regulatora GR

r_R - red astatizma regulatora G_R

$$G(s) = G_{\prime}(s) G_R(s) = K_0 K_R \frac{P_o(s) P_R(s)}{s^{r_0 + r_R} Q_0(s) Q_R(s)} = K \frac{P(s)}{s^r Q(s)}, \qquad P(0) = Q(0) = 1$$

$K = K_0 K_R$ - faktor pojačanja funkcije povratnog prenosa G

$r = r_0 + r_R$ - red astatizma funkcije povratnog prenosa G

Primer:

$$G(s)=\frac{s^2+2s+3}{s^2\left(s^4+3s^2+4\right)}=\frac{3\left(\frac{1}{3}s^2+\frac{2}{3}s+1\right)}{4s^2\left(\frac{1}{4}s^4+\frac{3}{4}s^2+1\right)}$$

$$=\frac{3}{4}\frac{\frac{1}{3}s^2+\frac{2}{3}s+1}{s^2\left(\frac{1}{4}s^4+\frac{3}{4}s^2+1\right)}$$

$$=\frac{3}{4}\frac{P(s)}{s^2Q(s)}$$

$$P(s)=\frac{1}{3}s^2+\frac{2}{3}s+1, \quad P(0)=1,$$

$$Q(s)=\frac{1}{4}s^4+\frac{3}{4}s^2+1, \; Q(0)=1,$$

$$K=3/4,$$

$$r=2$$

5.2.1. GREŠKA U STACIONOM STANJU

Ukoliko su ispunjeni uslovi važenja II granične teoreme Laplasa važi:

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} s \frac{1}{1+G(s)} R(s) + \lim_{s\to 0} s \frac{-G_0}{1+G} Z(s)$$

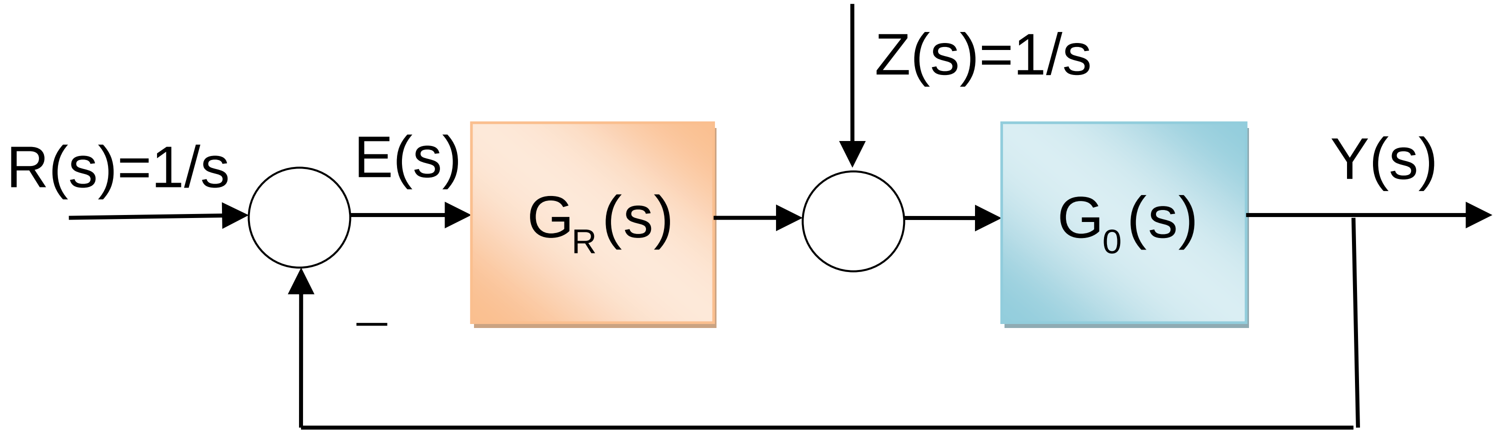
$$= e_S^R + e_S^Z$$

Greška u stacionom stanju zavisi od:

1.funkcija prenosa G_O i G

2.ulaznog signala i poremećaja

Ulaz: $r(t) = h(t) \Rightarrow R(s) = \frac{1}{s}$ Poremećaj: $z(t) = h(t) \Rightarrow Z(s) = \frac{1}{s}$



$$e_P(\infty) = \lim_{s\to 0} \frac{s}{1+G(s)}\frac{1}{s} + \lim_{s\to 0} \frac{-sG_0(s)}{1+G}\frac{1}{s} = \frac{1}{1+\lim\limits_{s\to 0} G(s)} + \frac{-\lim\limits_{s\to 0} G_0(s)}{1+\lim\limits_{s\to 0} G(s)}$$

$$= \frac{1}{1+K_P} + \frac{-K_{0P}}{1+K_P} = e_{SP}^{R} + e_{SP}^{Z}$$

gde su : $K_P = \lim\limits_{s\to 0} G(s)$, $K_{0P} = \lim\limits_{s\to 0} G_0(s)$

Poziciona statička greška usled pobude

$$e_{SP}^{R} = 1/(1+K_P)$$

Poziciono pojačanje (konstanta položaja) funkcije povratnog prenosa G:

$$K_P = \lim_{s\to 0} G(s)$$

$$K_P = \lim_{s\to 0} G(s) = \lim_{s\to 0} K \frac{P(s)}{Q(s)} \frac{1}{s^r} = K \lim_{s\to 0} \frac{1}{s^r} = \begin{cases} K = K_0 K_R, & r = r_0 + r_R = 0 \\ \infty, & r = r_0 + r_R > 0, \ \forall K \end{cases}$$

$$e_{SP}^{R} = \begin{cases} \dfrac{1}{1+K_0 K_R}, & r = r_0 + r_R = 0 \\ 0 \quad , & r = r_0 + r_R > 0 \end{cases}$$

Za $r = r_0 + r_R = 0 \;\Rightarrow\; e_{SP}^{R} = 1/(1+K_0 K_R)$.

Za $r = r_0 + r_R > 0 \;\Rightarrow\; e_{SP}^{R} = 0$ bez obzira na vrednost $K = K_0 K_R$.

Poziciona statička greška usled poremećaja

$$e_{SP}^{Z} = -K_{0P} / (1 + K_P)$$

Poziciono pojačanje objekta G_0:

$$K_{0P} = \lim_{s \to 0} G_0(s)$$

$$K_{0P} = \lim_{s \to 0} G_0(s) = \lim_{s \to 0} K_0 \frac{P_0(s)}{Q_0(s)} \frac{1}{s^{r_0}} = \lim_{s \to 0} K_0 \frac{1}{1} \frac{1}{s^{r_0}} = K_0 \lim_{s \to 0} \frac{1}{s^{r_0}} = \begin{cases} K_0, & r_0 = 0 \\ \infty, & r_0 > 0 \end{cases}$$

Poziciono pojačanje funkcije povratnog prenosa G:

$$K_P = \lim_{s \to 0} G(s) = \begin{cases} K = K_0 K_R, & r = r_0 + r_R = 0 \\ \infty, & r = r_0 + r_R > 0, \ \forall K \end{cases}$$

Poziciona statička greška usled poremećaja

$$e_{SP}^{Z}=-\lim_{s\to 0}\frac{G_O(s)}{1+G(s)}=-\lim_{s\to 0}\frac{K_O\dfrac{P_O(s)}{s^{r_O}Q_O(s)}}{1+K\dfrac{P(s)}{s^{r}Q(s)}}=-\lim_{s\to 0}\frac{K_O P_O(s)Q(s)}{Q_O(s)\left(s^{r}Q(s)+KP(s)\right)}s^{r-r_0}$$

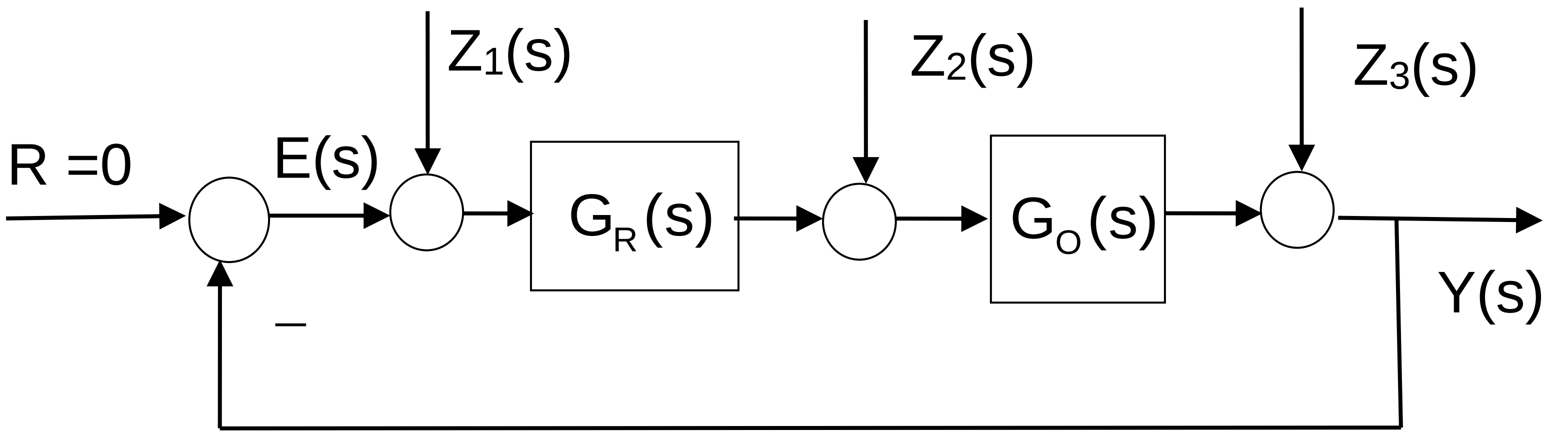
$$=-\lim_{s\to 0}\frac{K_O}{K}s^{r-r_0}=-\frac{K_O}{K_O K_R}\lim_{s\to 0}s^{r_0+r_R-r_0}=-\frac{1}{K_R}\lim_{s\to 0}s^{r_R}$$

$$e_{SP}^{Z}=\begin{cases}-\dfrac{1}{K_R}, & r_R=0\\ 0\ , & r_R>0\end{cases}$$

Slučaj $r_R=0 \quad\Rightarrow\quad e_{SP}^{Z}=-1/K_R.$

Slučaj $r_R>0 \quad\Rightarrow\quad e_{SP}^{Z}=0$ bez obzira na vrednost faktora pojačanja K i K_0.

Dejstvo više poremećaja, pobuda ukinuta



$$E^{Z}(s) = \underbrace{\left(-\frac{G(s)}{1+G(s)} Z_1(s)\right)}_{E^{Z_1}(s)} + \underbrace{\left(-\frac{G_O(s)}{1+G(s)} Z_2(s)\right)}_{E^{Z_2}(s)} + \underbrace{\left(-\frac{1}{1+G(s)} Z_3(s)\right)}_{E^{Z_3}(s)}$$

$$
\begin{aligned}
e_{SP}^{Z}(\infty) &= \lim_{s\to 0} sE_P^{Z}(s) \\
&= \lim_{s\to 0} sE_P^{Z_1}(s) + \lim_{s\to 0} sE_P^{Z_2}(s) + \lim_{s\to 0} sE_P^{Z_3}(s) \\
&= -\lim_{s\to 0} s\frac{G(s)}{1+G(s)}\frac{1}{s} - \lim_{s\to 0} s\frac{G_O(s)}{1+G(s)}\frac{1}{s} - \lim_{s\to 0} s\frac{1}{1+G(s)}\frac{1}{s} \\
&= -\lim_{s\to 0}\frac{G(s)}{1+G(s)} - \lim_{s\to 0}\frac{G_O(s)}{1+G(s)} - \lim_{s\to 0}\frac{1}{1+G(s)} \\
&= \left(-\frac{\lim_{s\to 0} G(s)}{1+\lim_{s\to 0} G(s)}\right) + \left(-\frac{\lim_{s\to 0} G_O(s)}{1+\lim_{s\to 0} G(s)}\right) + \left(-\frac{1}{1+\lim_{s\to 0} G(s)}\right) \\
&= e_{SP}^{Z_1}(\infty) + e_{SP}^{Z_2}(\infty) + e_{SP}^{Z_3}(\infty)
\end{aligned}
$$

Greška poremećaja koji deluje ispred regulatora:

$$e_{SP}^{Z_1}(\infty) = -\lim_{s\to 0} \frac{G(s)}{1+G(s)} = -\frac{K\frac{P(s)}{Q(s)}\frac{1}{s^r}}{1+K\frac{P(s)}{Q(s)}\frac{1}{s^r}} = -\lim_{s\to 0}\frac{K\frac{1}{s^r}}{1+K\frac{1}{s^r}} = -\lim_{s\to 0}\frac{K}{s^r+K} = -1 \neq 0$$

Greške poremećaja koji deluju iza regulatora:

$$e_{SP}^{Z_2}(\infty) = -\lim_{s\to 0}\frac{G_O(s)}{1+G(s)} = -\frac{1}{K_R}\lim_{s\to 0} s^{r_R} = 0, \quad r_R > 0$$

$$e_{SP}^{Z_3}(\infty) = -\lim_{s\to 0}\frac{1}{1+G(s)} = \frac{1}{1+K\lim_{s\to 0}\frac{1}{s^r}} = 0, \quad r = r_0 + r_R > 0$$

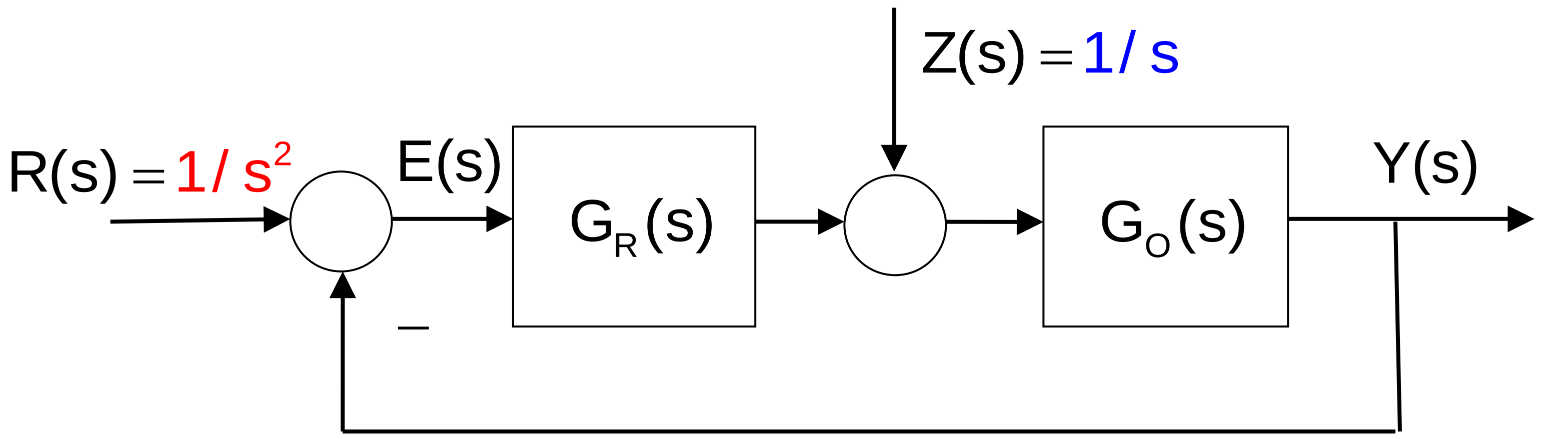
Ukupna greška svih poremećaja:

$$e_{SP}^{Z}(\infty) = -1-0-0 = -1 \neq 0$$

Zaključak. Da bi $e_{SP}^{Z}(\infty) = 0 \quad \Rightarrow$ treba regulator sa $r_R > 0$ postaviti ispred mesta dejstva poremećaja.

5.2.2. BRZINSKA KONSTANTA

Ulaz: $r(t) = th(t) \Rightarrow R(s) = \dfrac{1}{s^2}$ Poremećaj: $z(t) = h(t) \Rightarrow Z(s) = \dfrac{1}{s}$



Brzinska statička greška:

$$e_v(\infty) = \lim_{s\to 0} s \frac{1}{1+G(s)} \frac{1}{s^2} - \lim_{s\to 0} s \frac{G_O(s)}{1+G} \frac{1}{s} = \frac{1}{\lim\limits_{s\to 0} sG(s)} - \frac{\lim\limits_{s\to 0} G_O(s)}{1+\lim\limits_{s\to 0} G}$$

$$= \frac{1}{K_V} - \frac{K_{OP}}{1+K_P} = e_{SV}^R + e_{SP}^Z$$

Brzinska konstanta sistema ili brzinsko pojačanje funkcije povratnog prenosa.

$$K_V = \lim_{s \to 0} sG(s)$$

$$K_V = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{KP(0)}{Q(0)} \frac{1}{s^{r-1}} = \begin{cases} 0, & r = r_0 + r_R = 0 \\ K = K_0 K_R, & r = r_0 + r_R = 1 \\ \infty, & r = r_0 + r_R > 1, \quad \forall K \end{cases}$$

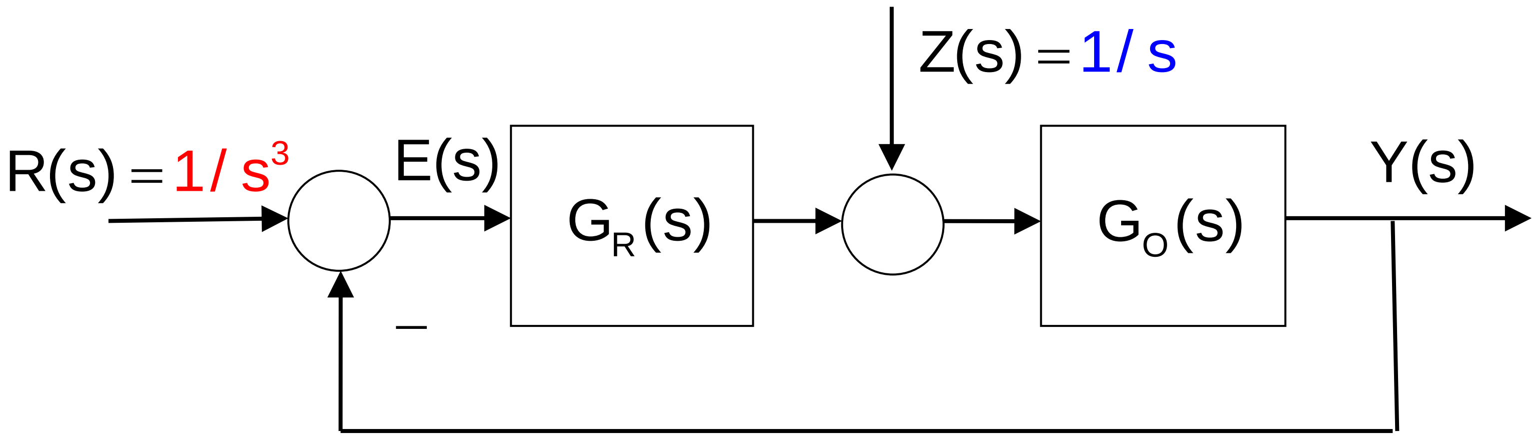
Brzinska statička greška:

$$e_{SP}^{R} = \frac{1}{K_V} = \begin{cases} \infty, & r = r_0 + r_R = 0 \\ \frac{1}{K} = \frac{1}{K_0 K_R}, & r = r_0 + r_R = 1 \\ 0, & r = r_0 + r_R > 1, \forall K \end{cases}$$

5.2.3. KONSTANTA UBRZANJA

Ulaz: $r(t)=\frac{1}{2}t^2h(t)\Rightarrow R(s)=\frac{1}{s^3}$

Poremećaj: $z(t)=h(t)\Rightarrow Z(s)=\frac{1}{s}$



Statička greška ubrzanja:

$$e_A(\infty)=\lim_{s\to 0} s\frac{1}{1+G(s)}\frac{1}{s^3}-\lim_{s\to 0} s\frac{G_O(s)}{1+G(s)}\frac{1}{s}=\frac{1}{\lim\limits_{s\to 0} s^2G(s)}-\frac{\lim\limits_{s\to 0} G_O(s)}{1+\lim\limits_{s\to 0} G(s)}$$

$$=\frac{1}{K_A}-\frac{K_{OP}}{1+K_P}=e_{SA}^R+e_{SP}^Z$$

Konstanta ubrzanja sistema ili pojačanje ubrzanja funkcije povratnog prenosa

$$K_A = \lim_{s \to 0} s^2 G(s)$$

$$K_A = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} \frac{KP(0)}{Q(0)} \frac{1}{s^{r-2}} = \begin{cases} 0, & r = r_0 + r_R = 0,1 \\ K = K_0 K_R, & r = r_0 + r_R = 2 \\ \infty, & r = r_0 + r_R > 2 \end{cases}$$

Statička greška ubrzanja:

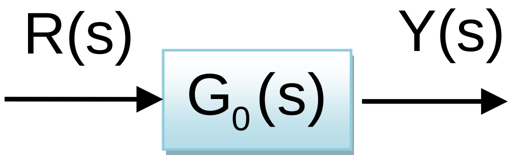
$$e_{SP}^{R} = \frac{1}{K_A} = \begin{cases} \infty, & r = r_0 + r_R = 0,1 \\ \dfrac{1}{K} = \dfrac{1}{K_0 K_R}, & r = r_0 + r_R = 2 \\ 0, & r = r_0 + r_R > 2 \end{cases}$$

GREŠKA U STACIONARNOM STANJU SISTEMA SA POVRATNOM SPREGOM U FUNKCIJI POBUDE I REDA ASTATIZMA

<table>
<tr><td colspan="2"></td><td colspan="4">referenca</td></tr>
<tr><td colspan="2"></td><td>$h(t)$</td><td>$th(t)$</td><td>$\frac{1}{2}t^2h(t)$</td><td>$\cdots$</td></tr>
<tr><td rowspan="4">astatizam reda r</td><td>$r = r_0 + r_R = 0$</td><td>$e_{SP}^R = \frac{1}{1+K_P}$</td><td>$e_{SV}^R = \infty$</td><td rowspan="2">$e_{SA}^R = \infty$</td><td rowspan="3">$e_{S\dots}^R = \infty$</td></tr>
<tr><td>$r = r_0 + r_R = 1$</td><td rowspan="3">$e_{SP}^R = 0$</td><td>$e_{SV}^R = \frac{1}{K_V}$</td></tr>
<tr><td>$r = r_0 + r_R = 2$</td><td rowspan="2">$e_{SV}^R = 0$</td><td>$e_{SA}^R = \frac{1}{K_A}$</td></tr>
<tr><td>$\vdots$</td><td>$e_{SA}^R = 0$</td><td></td></tr>
</table>

Primer 5.1. Objekat: $G_0(s)=\dfrac{K_0}{T_0 s+1}$, Poziciona greška $e_{SP}^R=?$

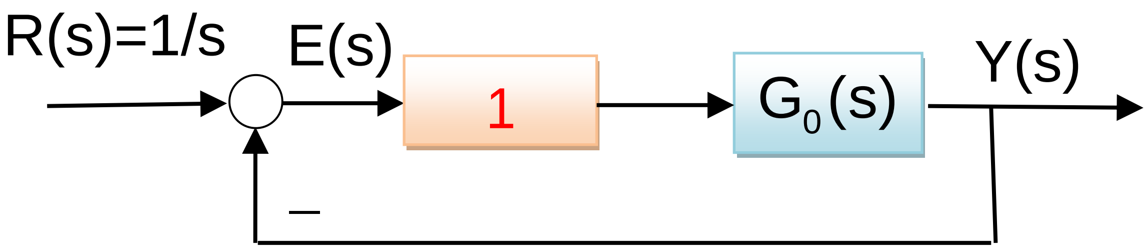
Za sistem bez povratne sprege je



$$e_{SP}^R=1-G_0(0)=1-K_P=1-K_0$$

$e_{SP}^R=0$ ako je $K_0=1$ (kalibracija)

Za sistem sa povratnom spregom, sa $G_R(s)=1$

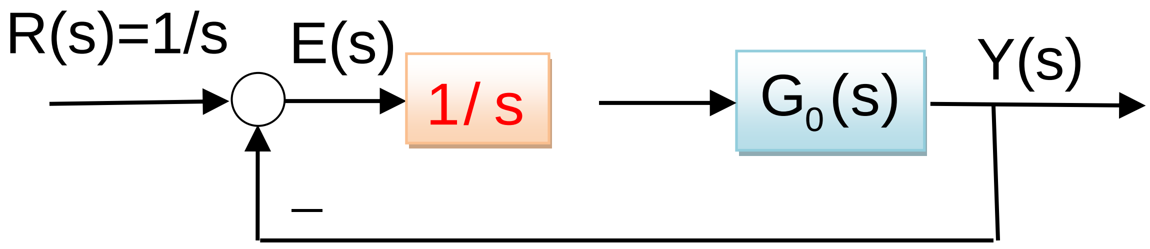


$$e_{SP}^R=\frac{1}{1+G_0(0)\cdot 1}=\frac{1}{1+K_0},$$

$e_{SP}^R\to 0$ kada $K_0\to\infty$

Usvojimo $K_0=100$ $\Rightarrow$ $e_{SP}^R=\dfrac{1}{1+100}=0.0099\approx 0\Rightarrow e_{SP}^R[\%]\approx 1\%$

Za sistem sa povratnom spregom i $G_R(s) = 1/s$ važi



$$K_P = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K_0}{s\left(T_0 s + 1\right)} = \infty$$

$$\Rightarrow e_{SP}^R = \frac{1}{K_P} = 0 \qquad \forall K_0$$

Neka se K_0 smanji za 10%

Sistem bez povratne sprege: $K_0 = 1 \quad \Rightarrow \quad \hat{K}_0 = 0.9$

$$e_{SP}^R = 1 - 0.9 = 0.1 \quad \Rightarrow \quad e_{SP}^R[\%] = 10\%$$

drastična promena u e_{SP}^R sa 0% na 10%

Sistem sa povratom spregom i $G_R(s) = 1$, $K_0 = 100 \Rightarrow \hat{K}_0 = 90$

$$e_{SP}^R = \frac{1}{1+\hat{K}_0} = \frac{1}{1+90} = \frac{1}{91} \quad \Rightarrow \quad e_{SP}^R[\%] \approx 1.1\%$$

neznatna promena u e_{SP}^R sa 1% na 1.1%

Za sistem sa povratnom spregom i $G_R(s) = 1/s$

$$K_P = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{K_O}{s(T_0 S + 1)} = \infty \Rightarrow e_{SP}^R = \frac{1}{K_P} = 0 \quad \forall K_0,$$

greška je jednaka 0

5.3. OČITAVANJE KONSTANTI GREŠKE SA BODEOVIH DIJAGRAMA

$$K_P = \lim_{s\to 0} G(s)$$
$$K_V = \lim_{s\to 0} sG(s)$$
$$K_A = \lim_{s\to 0} s^2 G(s)$$

Za $s = j\omega$

$$K_P = \lim_{\omega\to 0} G(j\omega) = \lim_{\omega\to 0} \frac{KP(j\omega)}{Q(j\omega)} = K$$ ako je red astatizma r = 0

$$K_V = \lim_{\omega\to 0} (j\omega) G(j\omega) = \lim_{\omega\to 0} (j\omega) \frac{KP(j\omega)}{j\omega Q(j\omega)} = K$$ ako je red astatizma r = 1

$$K_A = \lim_{\omega\to 0} (j\omega)^2 G(j\omega) = \lim_{\omega\to 0} (j\omega)^2 \frac{KP(j\omega)}{(j\omega)^2 Q(j\omega)} = K$$ ako je red astatizma r = 2

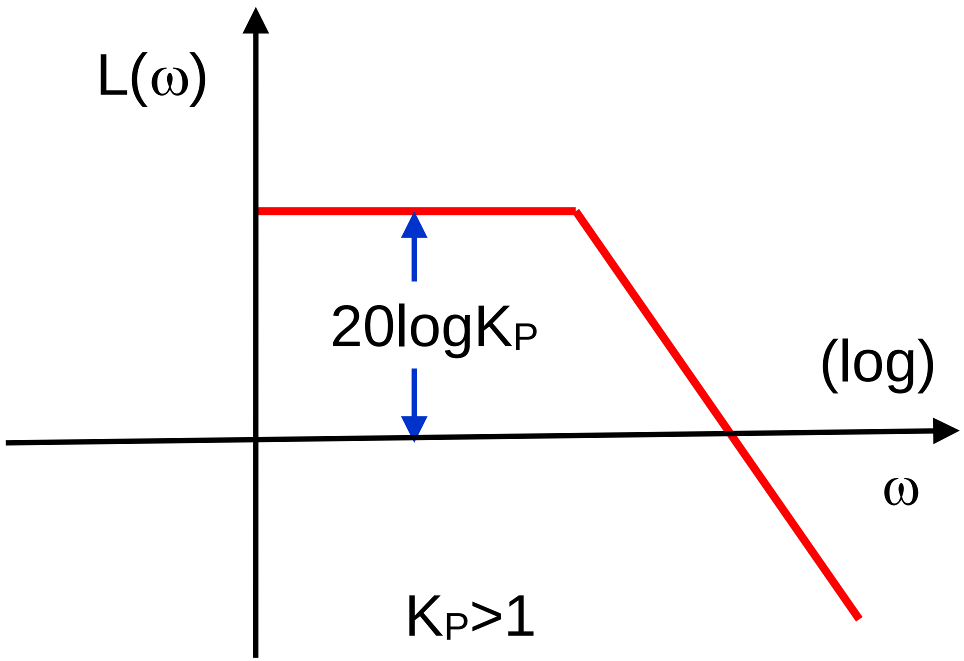
5.3.1. KONSTANTA POLOŽAJA K_P

$$\omega \to 0$$

$$K_P = \lim_{\omega \to 0} G(j\omega) = \lim_{\omega \to 0} \frac{KP(j\omega)}{Q(j\omega)} = K$$ ako je red astatizma r = 0

Početni segment amplitudne karakteristike sistema bez astatizma ($r = 0$) je

$$L(\omega) \approx 20\log\left|\frac{KP(j0)}{Q(j0)}\right| = 20\log K_P, \text{ za } \omega \approx 0$$



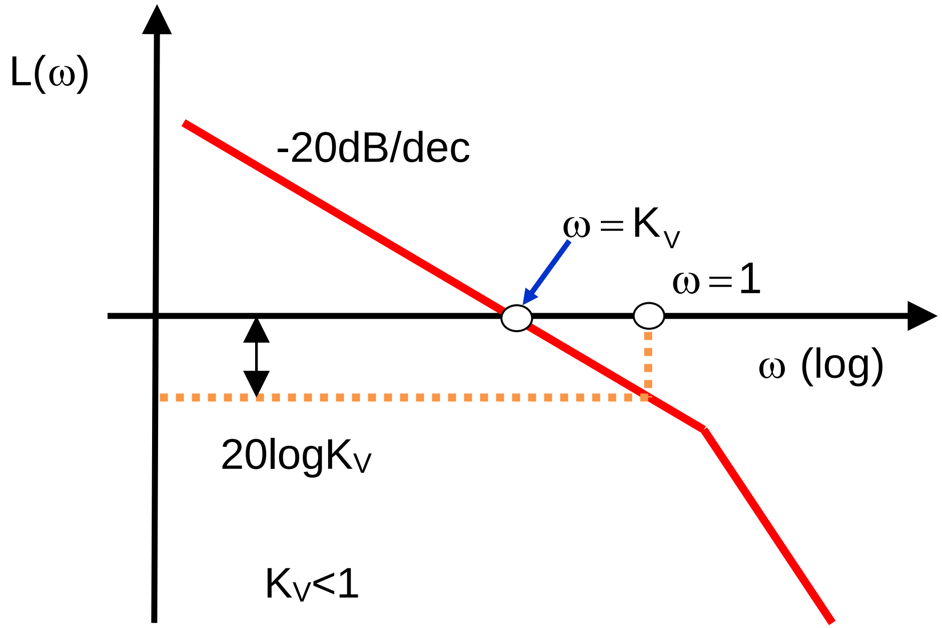
5.3.2. BRZINSKA KONSTANTA K_V

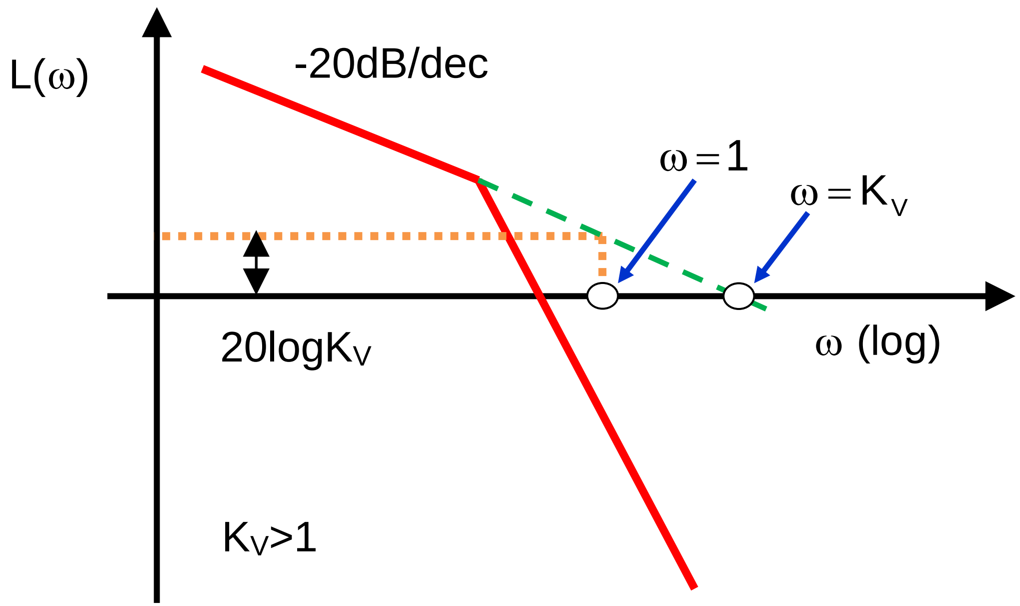
$\omega \to 0,\ K_V = \lim\limits_{\omega\to 0}(j\omega)G(j\omega) = \lim\limits_{\omega\to 0}(j\omega)\dfrac{KP(j\omega)}{j\omega Q(j\omega)} = K$, ako je astatizam r = 1

Početni segment amplitudne karakteristike sistema sa astatizmom prvog reda ($r = 1$) je

$$L(\omega) \approx 20\log\left|K_V / j\omega\right| = 20\log K_V - 20\log\omega, \text{ za } \omega \approx 0.$$

Za $20logK - 20log\omega = 0 \quad \Rightarrow \quad K = K_V = \omega.$





5.3.3. KONSTANTA UBRZANJA K_A

$$\omega \to 0,\ K_A = \lim_{\omega\to 0}(j\omega)^2 G(j\omega) = \lim_{\omega\to 0}(j\omega)^2 \frac{KP(j\omega)}{(j\omega)^2 Q(j\omega)} = K$$, red astatizma r = 2

Početni segment amplitudne karakteristike sistema sa astatizmom drugog reda ($r = 2$) je

$$L(\omega) \approx 20\log\left|K_A/(j\omega)^2\right| = 20\log K_A - 40\log\omega,\ \text{za}\ \omega \approx 0.$$

Za $20logK - 40log\omega = 0 \quad \Rightarrow \quad K = K_A = \omega^2$.